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ASSESSMENT OF THE RUIN PROBABILITIES

Abstract*:   In this paper, we analyze the ruin probability for some risk models, which is the probability that an insurer will face ruin in finite time when the insurer starts with initial reserve and is subjected to independent and identically distributed claims over time. The ideal is as we are able to come up with closed form solutions for the infinite horizon ruin probability and the finite horizon ruin probability. But, the cases where this is possible are few; therefore we must make approximations of ruin probability. In this paper, we insist on the discrete time insurance model and on the diffusion approximation and so-called "corrected diffusion approximation (CDA)". We analyze the ruin probability with respect to: the parameters of the individual claim distribution and the intensity parameter of the number of claims process. Ruin theory with debit and credit interest has received considerable attention in recent years. In this line, we consider a perturbed risk model in which a current premium rate will be adjusted in any period (usually year) in which there are no losses and any surplus available at the beginning of the period is reinvested. Also, we analyze and the inverse problem: to determine the initial reserve when it is given the ruin probability.*

Keywords: *Brownian motion, corrected diffusion approximation, risk process, ruin probability, surplus process.*

JEL Classification: C020, G220, G320

1. Introduction

The actuarial risk model has two main components: one characterizing the frequency of events and another describing the size (or severity) of gain or loss resulting from the occurrence of an event. The stochastic nature of both, the incidence and severity of claims, has an essential role for the set up of a realistic model. In

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examining the nature of the risk associated with a portfolio of business, it is often of interest to assess how the portfolio may be expected to perform over an extended period of time. One approach concerns the use of ruin theory. Ruin theory is concerned with the excess of the income (with respect to a portfolio of business) over the outgo, or claims paid. This quantity, referred to as insurer's surplus, varies in time. Specifically, ruin is said to occur if the insurer's surplus reaches a specified lower bound. One measure of risk is the probability of a suchlike event, clearly reflecting the volatility inherent in the business. This probability is called ruin probability. It can serve as a useful tool in long range planning for the use of insurer's funds. The company receives a certain amount of premium to cover its liabilities. The company is assumed to have a certain initial capital (risk reserve) at its disposal. One important problem in risk theory is to investigate the ruin probability, i.e. the probability that the risk business ever becomes negative. The ideal is as we are able to come up with closed form solutions for the infinite horizon ruin probability and the finite horizon ruin probability. But, the cases where this is possible are few; therefore we must make approximations of ruin probability. There are various ways to model aggregate claims distributions, the time evolution of the reserves of an insurance company and its claim surplus process and to define the probability of ruin. The idea behind the diffusion approximation is to first approximate the claim surplus process by a Brownian motion with drift by matching the two first moments. Since Brownian motion is skip-free, the idea to replace the risk process by a Brownian motion ignores the presence of the overshoot and other things. In this paper, we insist on ruin probability of discrete-time surplus process, and on the diffusion approximations of ruin probability.

In the classical risk model, the premium rate c is a fixed constant that satisfies a positive security loading condition, namely, $c > 0$, and the premium rate is irrespective of the claim experience. However, the premium rate in practice, especially for auto-insurances, is often adjusted according to the claim experience. The assumption that the premium rate keeps constant is very restrictive in practice. The models from ruin theory with debit and credit interest and with perturbed risk have received considerable attention in recent years. Also, multi-dimensional risk theory has gained a lot of attention in the past few years mainly due to the complexity of the problems and the lack of closed-form results even under very basic model assumptions.

One of the main questions relating to the operation of an insurance company is the calculation of the probability of ruin, and the probability of ruin before time T. The theory of martingales provides a quick way of calculating the risk of an insurance company. If $\left(X_n : n \geq 1 \right)$ be a sequence of independent and identically distributed random variables, let $S = \left(S_n : n \geq 0 \right)$ be its associated random walk (so that $S_0 = 0$

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and $S_n = X_1 + \cdots + X_n$ for $n \geq 1$), and suppose that S with drift μ. It is interesting to develop the high accuracy approximations to the distribution of the maximum random variable $M = \max\{S_n : n \geq 0\}$. For $u > 0$, $\{M > u\} = \{\tau(u) < \infty\}$, where $\tau(u) = \inf\{n \geq 1 : S_n > u\}$, so that computing the tail of M is equivalent to computing a level crossing probability for the random walk S. In insurance risk theory, $P(\tau(u) < \infty)$ is the probability that an insurer will face ruin in finite time when the insurer starts with initial reserve u and is subjected to independent and identically distributed claims over time. One important approximation holds as $\mu \downarrow 0$. This asymptotic regime corresponds in risk theory to the setting in which the safety loading is small. In this case, the approximation $P(M > u) \approx \exp(-2\mu \cdot u / \sigma^2)$ is valid, where $\sigma^2 = Var(X_1)$. Because the right hand side is the exact value of the level crossing probability for the natural Brownian approximation to the random walk S, it is often called the diffusion approximation to the distribution of M.

Also, in this paper, we consider a time dependent risk model for the surplus of an insurer, in which the current premium will be adjusted after a year without losses and the available amount of money is reinvested. At the same time, we also want to derive an equation satisfied by the survival probability and to determine the risk reserve.

The remainder of this paper is organized as follows. A brief literature review is given in Section 2. Section 3 presents a introduction in the models of surplus process and more detailed the assessment of the ruin probabilities. Finally, in Section 4, we exemplify our methods with numerical results starting our models from data of annual report on 2012 of two Romanian Insurance Company[1] and we present the paper concludes with some comments.

2. Literature Review

Recently, several new risk models have been proposed in the specialized literature, in which the premium income of an insurer is uncertain and depends on some random components in the surplus of an insurer. For example, Dufresne and Gerber (1991) added a diffusion to the classical compound Poisson surplus process. The diffusion

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[1] ABC Asigurări Reasigurări SA (ABCAR) and SC Generali România Asigurare Reasigurare SA (GRAR)

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describes an uncertainty of the aggregate premium income or an additional uncertainty to the aggregate claims. A time-dependent premium risk model can be found in Asmussen (2000), in which premium rates are adjusted continuously according to the current level of an insurer's surplus. Albrecher and Asmussen (2006) investigated an adaptive premium that is dynamically adjusted according to the overall claim experience. In addition, the dependence between other components in risk models was also studied. For instance, Albrecher and Boxma (2004) considered a dependent risk model in which the Poisson arrival rate of the next claim is determined by the previous claim size. They extended their model to a Markov-dependent risk model in which both arrival rates and claim size distributions are determined by the state of an underlying continuous-time of Markov chain type. Furthermore, Albrecher and Teugels (2006) have studied the risk models with dependence between inter-claim times and claim sizes. Significant results have been achieved in models in which claims occur according to a Poisson process; see for e.g. Cai (2007), Zhu and Yang (2008), Mitric and Sendova (2010), Asmussen and Albrecher (2010), Mitric, Bădescu and Stanford (2012) and the references therein. Meanwhile, the renewal risk model under such assumptions has been studied much less frequently in the literature during this period. One notable contribution is Konstantinides et al. (2010), where the authors present asymptotic results for the infinite time absolute ruin probability. The main focus of the paper of Mitric, Bădescu and Stanford (2012) was the analysis of the Gerber–Shiu discounted penalty function (Gerber and Shiu, 1998), starting from a general non-renewal risk model with constant force of interest. They presented a general methodology that leads to a tractable analytical solution for the Gerber–Shiu function (with a penalty that depends on the deficit only), with coefficients that are obtained in a recursive fashion. Moreover, they obtained closed form solutions for the absolute ruin probabilities and the deficit at the absolute ruin, extending the results obtained under the classical case with exponential claim amounts.

For the diffusion approximation, the idea to replace the risk process by a Brownian motion ignores the presence of the overshoot and other things. Siegmund (1979) proposed a so-called corrected diffusion approximation (CDA) that reflects information in the increment distribution beyond the mean and variance. The objective of the corrected diffusion approximation is to take this and other deficits into consideration. The set-up is the exponential family of compound risk processes with parameters. Blanchet and Glynn (2006) developed this method to the full asymptotic expansion initiated by Siegmund.

Fu K. A. and Ng C.Y.A. (2014) consider a continuous-time renewal risk model, in which the claim sizes and inter-arrival times form a sequence of independent and identically distributed random pairs, with each pair obeying a dependence structure. They suppose that the surplus is invested in a portfolio whose return follows a Lévy

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process. When the claim-size distribution is dominatedly-varying tailed, they obtained asymptotic estimates for the finite- and infinite horizon ruin probabilities.

3. Surplus Process and Assessment of the Ruin Probabilities

We are interested in the surplus process $\{U(t), t \geq 0\}$ in continuous-time (or its discrete-time version, $\{U_t, t = 0,1,\ldots\}$), which measures the surplus of the portfolio at time t. We begin at time zero with $U(0) = u$ (or in discrete case $U_0 = u$), the initial surplus (initial reserve, risk reserve). The surplus at time t is $U(t) = u + P(t) + A(t) - S(t)$ (or in discrete-time version $U_t = u + P_t + A_t - S_t$), where $\{P(t), t \geq 0\}$ (in discrete time $\{P_t, t = 0,1,\ldots\}$) is the premium process which measures all premiums collected up to time t, $\{S(t), t \geq 0\}$ (in discrete time $\{S_t, t = 0,1,\ldots\}$) is the loss process, which measures all losses paid up to time t, and $\{A(t), t \geq 0\}$ (in discrete time $\{A_t, t = 0,1,\ldots\}$) is the earning process, which measures all earnings of investment income up to time t. We make the following assumption: $P(t)$ (or P_t) may depend on $S(r)$ (or S_r) for $r < t$ (for example, dividends based on favorable past loss experience may reduce the current premium).

Definition 3.1 The continuous-time, finite-horizon survival probability is given by $\Phi(u, \tau) = P\big(U(t) \geq 0 \text{ for all } 0 \leq t \leq \tau \big| U(0) = u\big)$, the continuous-time, infinite-horizon survival probability is given by $\Phi(u) = P\big(U(t) \geq 0 \text{ for all } t \geq 0 \big| U(0) = u\big)$, the discrete-time, finite-horizon survival probability is given by $\tilde{\Phi}(u, \tau) = P\big(U_t \geq 0 \text{ for all } t = 0,1,\ldots,\tau \big| U_0 = u\big)$, the discrete-time, infinite-horizon survival probability is given by $\tilde{\Phi}(u) = P\big(U_t \geq 0 \text{ for all } t = 0,1,\ldots \big| U_0 = u\big)$, the continuous-time, finite-horizon ruin probability is given by $\Psi(u, \tau) = 1 - \Phi(u, \tau)$, the continuous-time, infinite-horizon ruin probability is given by $\Psi(u) = 1 - \Phi(u)$, the

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discrete-time, finite-horizon ruin probability is given by $\tilde{\Psi}(u,\tau)=1-\tilde{\Phi}(u,\tau)$, and the discrete-time, infinite-horizon ruin probability is given by $\tilde{\Psi}(u)=1-\tilde{\Phi}(u)$.

A) The general mathematical model of an insurance risk in continuous-time is composed of the following objects:

a) A sequence $\{X_i\}_{i=1,2,3,\ldots}$ of independent and identically distributed random variables, having the common distribution function F and a finite mean m. The random variable (r.v.) X_i is the cost of the ith individual claim.

b) The stochastic process $N=\{N(t);t\geq 0\}$, $N(t)$ is the number of claims paid by the company in the time interval $[0,t]$. The counting process N and the sequence $\{X_i\}$ are independent objects.

$$S(t)=\sum_{i=1}^{N(t)}X_i . \tag{1}$$

The total amount of claims in $[0,t]$ is $S(t)=\sum_{i=1}^{N(t)}X_i$.

The risk process $Y=\{Y(t);t\geq 0\}$ is defined by $Y(t)=c\cdot t-S(t)$, $\qquad$ (2)

where $c>0$ is the constant premium rate per unit time.

Thus, the insurer's surplus at time t is $\quad U(t)=u+Y(t)$, $\qquad$ (3)

where $u=U(0)$ is the initial capital. Also $\{U(t);t\geq 0\}$ is the risk process.

When the premium rate is not a constant, we obtain a generalized model. Thus, if the premium at the moment t is function $c(t)$, then $Y(t)=\int_0^t c(x)dx-S(t)$ $\qquad$ (4)

We denote $m_k=E\left[X_i^k\right], k=1,2,3,\ldots,$ and $M_X(\tau)=E\left[e^{\tau\cdot X}\right]$ the moment generating function (mgf) of the random variable X. Note that $m=m_1$.

We consider that there exists a constant η (average amount of claim per unit time) such that $\quad \dfrac{1}{t}\sum_{i=1}^{N(t)}X_i \xrightarrow[t\to\infty]{a.s.} \eta$. $\qquad$ (5)

We define the safety loading θ as the relative amount by which the premium rate c exceeds η, thus $\theta=\dfrac{c-\eta}{\eta}$. $\qquad$ (6)

In the classical risk model, the process N is a homogeneous Poisson process with intensity λ (arrival rate), so that the surplus process of an insurer is described by

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$$U(t) = u + c \cdot t - \sum_{i=1}^{N(t)} X_i \tag{7}$$

We will use the mean value principle in order to compute the net premiums, thus

$$c = (1+\theta) \cdot \lambda \cdot m. \tag{8}$$

In the continuous-time case, we have the following:

Definition 3.2 The ruin moment T is $\quad T = \inf\{t \geq 0 \mid U(t) < 0\}.$ (9)

Definition 3.3 The ruin probability with respect to initial reserve u and the safety loading θ is $\quad \Psi(u, \theta) = P\left(\inf_{t \geq 0} U(t) < 0 \mid U(0) = u, \, g(\theta) = c\right).$ (10)

The ruin probability as a function of initial reserve is

$$\Psi(u) = P\left(\inf_{t \geq 0} U(t) < 0 \mid U(0) = u\right) \tag{11}$$

In this case we have the following propositions.

Proposition 3.1. Assume that (5) holds.

i) If $\theta < 0$, then $\sup_{0 \leq t < \infty} (S(t) - c \cdot t) = \infty$ a.s. and hence $\Psi(u) = 1$ for all u.

ii) If $\theta > 0$, then $\sup_{0 \leq t < \infty} (S(t) - c \cdot t) < \infty$ a.s. and hence $\Psi(u) < 1$ for all sufficiently large u.

Let $h(r) = \int_0^\infty (e^{r \cdot x} - 1) dF(x)$ and $g(\theta) = (1+\theta) \cdot \lambda \cdot m$. The adjustment coefficient (or Lundberg exponent) R is the smallest positive solution of the equation:

$$\lambda \cdot h(r) - c \cdot r = 0. \tag{12}$$

Proposition 3.2. If the adjustment coefficient R exists, then:

a) the ruin probability is $\Psi(u, \theta) = e^{-R \cdot u} \cdot \left(E\left[e^{R \cdot S(\tau)}\right]\right)^{-1},$ (13)

where $S(\tau) = (-C(\tau) \mid \tau < \infty))$ represents the severity of the loss at the moment of ruin;

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b) (Cramer's asymptotic ruin formula)

$$\Psi(u,\theta) \sim C \cdot e^{-R \cdot u}, \ u \to \infty, \tag{14}$$

where $C = \dfrac{m \cdot \theta}{M_X^{'}(R) - m \cdot (1 + \theta)}$.

Corollary: If the individual claim follows an exponential distribution with

parameter α , then $\Psi(u,\theta) = \dfrac{\lambda}{\alpha \cdot g(\theta)} \cdot e^{-\left(\alpha - \frac{\lambda}{g(\theta)}\right) \cdot u}$.

We will focus on an approach of estimating a risk process and the ruin probability using the Brownian motion. In some particular cases, it is obtained the diffusion approximation, which can be derived from approximating the risk process with a Wiener process (Brownian motion) with drift. It regards the way the surplus process $\{Y(t)\}_t$ based on the compound Poisson process is related to the Wiener process. Take a limit of the process $\{Y(t)\}_t$ as the expected number of downward jumps becomes large and simultaneously the size of the jumps becomes small (i.e. $\lambda \to \infty$ and $\alpha \to 0$, where the jump size is $X = \alpha \cdot V$, so that V has fixed mean and variance). Because the Wiener process with drift is characterized by the infinitesimal mean μ and infinitesimal variance σ^2, we impose the mean and variance functions to be the same for both processes. Thus, $\mu = c - \lambda \cdot m$ and $\sigma^2 = \lambda m_2$. The mgf of $Y(t)$ is $M_{Y(t)}(\tau) = \exp\{t \cdot (\tau \cdot c + \lambda \cdot (M_X(-\tau) - 1))\}$. Therefore

$$\lim_{\alpha \to 0} M_{Y(t)}(\tau) = \exp\left(\tau \cdot \mu \cdot t + \frac{\tau^2}{2} \cdot \sigma^2 \cdot t\right),$$ which is the mgf of the normal

distribution $N(\mu t, \sigma^2 t)$. Let $\{W(t), t \geq 0\}$ denote the Wiener process with mean function $\mu \cdot t$ and variance function $\sigma^2 \cdot t$. We consider the probability of ruin in a time interval $(0, \tau)$. Let $T = \inf_{t>0}\{t \mid u + W(t) < 0\}$. The probability of ruin before τ is

$\Psi(u, \tau) = P(T < \tau) = P\left(\inf_{0 < t < \tau} W(t) < -u\right).$

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Proposition 3.3. The probability of ruin before τ is

$$\Psi(u,\tau) = \Phi\left(-\frac{u+\mu\cdot\tau}{\sigma\cdot\sqrt{\tau}}\right) + e^{-2\frac{\mu}{\sigma^2}\cdot u}\cdot\Phi\left(-\frac{u-\mu\cdot\tau}{\sigma\cdot\sqrt{\tau}}\right). \tag{15}$$

Letting $\tau\rightarrow\infty$, the ultimate ruin probability is $\Psi(u) = e^{-2\frac{\mu\cdot u}{\sigma^2}} = e^{-2\frac{m}{m_2}\cdot\theta\cdot u}$ (i.e. the diffusion approximation).

Corollary: The probability density function of the time length until ruin is given by

$$f_T(\tau) = \frac{u}{\sigma\cdot\sqrt{2\pi}}\cdot\tau^{-\frac{3}{2}}\cdot e^{-\frac{(u-\mu\cdot\tau)^2}{2\sigma^2\cdot\tau}}, \quad \tau > 0.$$

Hence to obtain the expected time until ruin, given that it occurs, the idea behind the diffusion approximation is to first approximate the claim surplus process by a Brownian motion with drift by matching the two first moments. Consider that the claim sizes are independent and identically distributed non-negative random variables with cumulative distribution function F and finite mean m and finite variance σ^2.

Thus the standard diffusion approximation is

$$\Psi(u) \approx \Psi_{DA}(u,\theta) = \exp\left(-2\cdot\theta\cdot u\cdot\frac{m}{m^2+\sigma^2}\right) \tag{16}$$

For light-tailed random walk problems Siegmund (1979) derived a correction which was adapted to ruin probabilities by Asmussen and Binswanger (1997). An alternative covering also certain heavy-tailed cases was given of Hogan (1986). The result will be an approximation of the type

$$\Psi(u) \approx \Psi_{CDA}(u,\theta) = \left(1+\frac{4\cdot\theta^2\cdot u\cdot m_1^2\cdot m_3}{3m_2^3}-\frac{2\cdot\theta\cdot m_1\cdot m_3}{3m_2^2}\right)\cdot\exp\left(-2\cdot\theta\cdot u\cdot\frac{m_1}{m_2}\right) \tag{17}$$

when $m_5 < \infty$, where m_i is the i-th moment of F (evident $m_1 = m$). It is so-called corrected diffusion approximation of the ruin probability. When F is the Uniform(0,b) distribution function we obtain: $\quad \Psi_{DA}(u,\theta) = \exp\left(-3\cdot\theta\cdot\frac{u}{b}\right)$ \hfill (18)

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$$\Psi_{CDA}(u,\theta) = \left(1 + \frac{9 \cdot \theta^2 \cdot u}{4 \cdot b} - \frac{3 \cdot \theta}{4}\right) \cdot \exp\left(-3 \cdot \theta \cdot \frac{u}{b}\right) \qquad (19)$$

In this formula we can normalize b.

Another risk model is the Sparre Andersen model. This satisfies the following hypotheses:

i) The claim sizes $\xi_1, \xi_2, \ldots$ form a sequence of i.i.d. random variables with common distribution function B that has a finite mean $\mu > 0$;

ii) Occurrence times $T_1, T_2, \ldots$ are independent of $\xi_n, n \geq 1$, hence inter-occurrence times $\Theta_1 = T_1$, $\Theta_n = T_n - T_{n-1}, n \geq 2$, are i.i.d. random variables independent of $\xi_n, n \geq 1$. We assume that $U_0 = u$ is an initial risk reserve, and that the insurance company receives a sum that equals c per unit time deterministically (i.e. the intensity of the gross risk premium $c > 0$).

Let U_n be the level of the risk process just after the nth payoff. Therefore, we have $U_n = U_{n-1} + c \cdot \Theta_n - \xi_n, n \geq 1$. Let $Y_n = \xi_n - c \cdot \Theta_n, n \geq 1$. By G we denote a

distribution function (d.f.) of a random variable Y_1, then $G(u) = \int_0^\infty B(u+y)dF(y)$,

where F is the d.f. of a random variable $c \cdot \Theta_1$. Let $\theta = \dfrac{E(c \cdot \Theta_1 - \xi_1)}{E\xi_1}$ be a relative

safety loading. We assume $0 < \theta < \infty$. We denote $S_0 = 0$, $S_n = \sum_{k=1}^n Y_k, n \geq 1$, and

the ruin moment $\tau(u) = \inf\{n > 0 : S_n > u\}$. Then, $\Psi(u,n) = P(\tau(u) \leq n)$ is the

probability of ruin before the nth payoff, and $\Psi(u) = P(\tau(u) < \infty)$ is the probability

of ruin in infinite time. Let $M = \sup_{n \geq 0} S_n$. In this case M is finite almost surely. Thus,

we have $\Psi(u) = P(M > u)$ and $\Psi(u,n) = P\left(\max_{k \leq n} S_k > u\right)$. We denote by G_I the

integrated tail distribution of G, i.e. $G_I(x) = \left(\int_0^x \bar{G}(y)dy\right) \cdot \left(\int_0^\infty \bar{G}(y)dy\right)^{-1}, x \geq 0$,

where $\bar{G} = 1 - G$. If G_I belongs to the sub exponential class S, then the approximation for the probability of ruin in infinite time is given by

$\Psi(u) \sim (\theta \cdot \mu)^{-1} \int_u^\infty \bar{G}(y)dy$ as $u \to \infty$.

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B)   We consider a discrete-time insurance model.

Let the increment in the surplus process in period (usually year) t be defined as $W_t = P_t + A_t - S_t$, where: P_t is the premium collected in the tth period, S_t is the losses paid in the tth period, A_t is any cash flow other than the premium and the payment of losses, the most significant cash flow is the earning of investment income on the surplus available at the beginning of the period. The surplus at the end of the tth period is then

$$U_t = U_{t-1} + P_t + A_t - S_t = u + \sum_{j=1}^{t} \left(P_j + A_j - S_j \right). \tag{20}$$

Let the assumption that, given U_{t-1}, the random variable W_t depends only upon U_{t-1} and not upon any other previous experience.

For the discrete-time insurance model, we evaluate the ruin probability using the method of convolutions. The calculation of ruin probability $\Psi(u,t) = P\left(U_t < 0 | U_0 = u\right)$ is recursively, using distribution of U_t. Suppose that we obtained the discrete probability function (pf) of nonnegative surplus U_{t-1}:

$f_j^{(t-1)} = P\left(U_{t-1} = u_j\right), j = 1, 2, \ldots, n$ where $u_j \geq 0, \forall j$. Then the ruin probability is

$\Psi(u, t-1) = P\left(U_{t-1} < 0 | U_0 = u\right)$. Let $g_{j,k} = P\left(W_t = w_{j,k} | U_{t-1} = u_j\right)$. Then to

obtain
$$\Psi(u,t) = \Psi(u, t-1) + \sum_{j=1}^{n} \sum_{w_{j,k}+u_j < 0} g_{j,k} \cdot f_j^{(t-1)} \tag{21}$$

and $P\left(U_t = a\right) = \sum_{j=1}^{n} \sum_{w_{j,k}+u_j=a} g_{j,k} \cdot f_j^{(t-1)}$.

We shall use $S_t = \sum_{i=1}^{n} X_i$, where n is the number of insurance contracts and X_i is claim of contract i. For S_t we shall use the bounds $sm = n \cdot \min_{1 \leq i \leq n} X_i$ and

$SM = n \cdot \max_{1 \leq i \leq n} X_i$. In particular case, when $X : \begin{pmatrix} 0 & S \\ 1-p & p \end{pmatrix}$, we have

$sm : \begin{pmatrix} 0 & n \cdot S \\ 1 - p^n & p^n \end{pmatrix}$ and $SM : \begin{pmatrix} 0 & n \cdot S \\ (1-p)^n & 1-(1-p)^n \end{pmatrix}$.

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Here, we analyze the inverse problem: to determine the initial reserve when it is given the ruin probability. The following notations will be used: u - the risk reserve, $u^{\min}$ - the minimum reserve risk, α - the accepted probability of ruin, θ - the safety loading factor of the risk premium, X_i, $i=1,2,...$ - the independent and identical distributed random variables describing the claims or losses, with expectation $E(X)=m$ and variance $Var(X)=\sigma^2$, n - the number of insurance contracts, $S_n = \sum_{i=1}^{n} X_i$ is the aggregate demand or claim. Calculating the risk premium on the basis of the mean value principle, the premium income or revenue $PRIM$ is $PRIM = (1+\theta) \cdot m \cdot n$. The condition the accepted ruin probability should fulfill is:

$$P(S_n - PRIM > u) \leq \alpha. \tag{22}$$

As $(S_n - m \cdot n > \theta \cdot m \cdot n + u) \subseteq (|S_n - m \cdot n| \geq \theta \cdot m \cdot n + u)$, using Chebyshev's

inequality, we get: $P(S_n - m \cdot n > \theta \cdot m \cdot n + u) \leq \dfrac{n \cdot \sigma^2}{(\theta \cdot m \cdot n + u)^2}$ (23)

From (22) and (23), it follows that:

$$u \geq \sigma \cdot \sqrt{\dfrac{n}{\alpha}} - \theta \cdot m \cdot n \quad \text{and} \quad u_{Cheb}^{\min} = \sigma \cdot \sqrt{\dfrac{n}{\alpha}} - \theta \cdot m \cdot n. \tag{24}$$

From the non-negativity condition of the reserve, we obtain an upper bound of the load

factor $\theta \leq \dfrac{\sigma}{m \cdot \sqrt{n \cdot \alpha}}$. As well, we have: $\max_{n} u_{Cheb}^{\min} = \dfrac{\sigma^2}{4\alpha \cdot \theta \cdot m}$, (25)

Let us denote $Z_n = \dfrac{S_n - n \cdot m}{\sigma \cdot \sqrt{n}}$, then, using (22), we obtain

$P\left(Z_n \leq \dfrac{\theta \cdot n \cdot m + u}{\sigma \cdot \sqrt{n}} \right) \geq 1 - \alpha$ and from the Central Limit Theorem (CLT), we get:

$$u \geq \sigma \cdot z_{1-\alpha} \cdot \sqrt{n} - \theta \cdot m \cdot n \quad \text{and} \quad u_{CLT}^{\min} = \sigma \cdot z_{1-\alpha} \cdot \sqrt{n} - \theta \cdot m \cdot n \tag{26}$$

where $z_{1-\alpha}$ is the cuantile of order $1-\alpha$ of the standard normal distribution, $\Phi(z_{1-\alpha}) = 1 - \alpha$. Here, we can use the main result of Schulte (2012) that the volume of the Poisson-Voronoi approximation behaves asymptotically like a Gaussian random

variable if the intensity of the Poisson point process goes to infinity. An alternative approach would be to apply the underlying general central limit theorem directly to the infinite Wiener–Itô chaos expansion, which gives a sum of an infinite number of expected values of products of multiple Wiener–Itô integrals as an upper bound.

Similarly as above, we obtained an upper bound of the load factor: $\theta \le \dfrac{\sigma \cdot z_{1-\alpha}}{m \cdot \sqrt{n}}$.

We also have: $\max\limits_{n} u_{CLT}^{\min} = \dfrac{\sigma^2 \cdot z_{1-\alpha}^2}{4 \cdot \theta \cdot m}$, $\hspace{3cm}$ (27)

4. Numerical illustration and conclusions

From 2012 Annual Report of Romanian Insurance Company (GRAR) we observe that exist insurance policies which produce the ruin: gross written premiums for RCA are 66,991,572 lei, but then gross indemnity payments for RCA are 123,099,067 lei, however gross written premiums for fire policies are 96,961,352 lei and gross indemnity payments for them are 13,909,967 lei.

We give a first scenario for the discrete-time model. Suppose that annual losses can assume the values 0, 2, 4, 8, and 10 monetary units (m.u.), with probabilities 0.3, 0.3, 0.2, 0.1, and 0.1, respectively and losses are paid at the end of the year. Further suppose that the initial surplus is 5 m.u., and a premium of 3 m.u. is collected at the beginning of each year. Interest is earned at 4% on any surplus available at the beginning of the year. In addition, a rebate of 0.5 u.m. is given in any year in which there are no losses. We want to determine the ruin probability at the end of each of the first three years. We using formula (21), the obtained results are gived in Table 1.

Table 1. The surplus process and the ruin probabilities

U_{t-1} (m.u.)	$f_j^{(t-1)}$	$\left(w_{j,1}, g_{j,1}\right)$ $\left(0;0.3\right)$	$\left(w_{j,2}, g_{j,2}\right)$ $\left(2;0.3\right)$	$\left(w_{j,3}, g_{j,3}\right)$ $\left(4;0.2\right)$	$\left(w_{j,4}, g_{j,4}\right)$ $\left(8;0.1\right)$	$\left(w_{j,5}, g_{j,5}\right)$ $\left(10;0.1\right)$
5	1	(7.82;0.3)	(6.32;0.3)	(4.32;0.2)	(0,32;0.1)	(<0;0.1)
$\Psi(5,1)=0.1$						

7.82	0.3	(10.7528; 0.09)	(9.2528; 0.09)	(7.2528; 0.06)	(3.2528; 0.03)	(1.2528; 0.03)
6.32	0.3	(9.1928; 0.09)	(7.6928; 0.09)	(5.6928; 0.06)	(1.6928; 0.03)	(<0;0.03)
4.32	0.2	(7.1128; 0.06)	(5.6128; 0.06)	(3.6128; 0.04)	(<0;0.02)	(<0;0.02)
0.32	0.1	(2.9528; 0.03)	(1.4528; 0.03)	(<0;0.02)	(<0;0.01)	(<0;0.01)
$\Psi(5, 2) = 0.21$						
10.7528	0.09	(13.802912; 0.027)	(12.302912; 0.027)	(10.302912; 0.018)	(6.302912; 0.009)	(4.302912; 0.009)
9.2528	0.09	(12.242912; 0.027)	(10.742912; 0.027)	(8.742912; 0.018)	(4.742912; 0.009)	(2.742912; 0.009)
7.2528	0.06	(10.162912; 0.018)	(8.662912; 0.018)	(6.662912; 0.012)	(2.662912; 0.006)	(0.662912; 0.006)
3.2528	0.03	(6.002912; 0.009)	(4.502912; 0.009)	(2.502912; 0.006)	(<0;0.003)	(<0;0.003)
1.2528	0.03	(3.922912; 0.009)	(2.422912; 0.009)	(0.422912; 0.006)	(<0;0.003)	(<0;0.003)
9.1928	0.09	(12.180512; 0.027)	(10.680512; 0.027)	(8.680512; 0.018)	(4.680512; 0.009)	(2.680512; 0.009)
7.6928	0.09	(10.620512; 0.027)	(9.120512; 0.027)	(7.120512; 0.018)	(3.120512; 0.009)	(1.120512; 0.009)
5.6928	0.06	(8.540512; 0.018)	(7.040512; 0.018)	(5.040512; 0.012)	(1.040512; 0.006)	(<0;0.006)
1.6928	0.03	(4.380512; 0.009)	(2.880512; 0.009)	(0.880512; 0.006)	(<0;0.003)	(<0;0.003)
7.1128	0.06	(10.017312; 0.018)	(8.517312; 0.018)	(6.517312; 0.012)	(2.517312; 0.006)	(0.517312; 0.006)

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5.6128	0.06	(8.457312; 0.018)	(6.957312; 0.018)	(4.957312; 0.012)	(0.957312; 0.006)	(<0;0.006)
3.6128	0.04	(6.377312; 0.012)	(4.877312; 0.012)	(2.877312; 0.008)	(<0;0.004)	(<0;0.004)
2.9528	0.03	(5.690912; 0.009)	(4.190912; 0.009)	(2.190912; 0.006)	(<0;0.003)	(<0;0.003)
1.4528	0.03	(4.130912; 0.009)	(2.630912; 0.009)	(0.630912; 0.006)	(<0;0.003)	(<0;0.003)
$\Psi(5,3)=0.26$					

For second scenario, we suppose ten policies with annual loss can assume the value 0.2 m.u. with probability 0.6, and loss is paid at the end of the year. Further suppose that the initial surplus is 1 m.u., and a premium of 0.5 m.u. is collected at the beginning of each year. Interest is earned at 4% on any surplus available at the beginning of the year. In addition, a rebate of 0.1 m.u. is given in any year in which there is no loss. We determine the down border of ruin probability ($db\Psi$) at the end of each of the first four years. Using *sm* we obtained the results in Table 2.

Table 2. The down border of the ruin probabilities

U_{t-1} (m.u.)	$f_j^{(t-1)}$	$\left(w_{j,1}, g_{j,1}\right)$ (0;0.993953383)	$\left(w_{j,2}, g_{j,2}\right)$ (2;0.006046617)
1	1	(1.56;0.993953383)	(<0;0.006046617)
$db\Psi(1,1)=0.006046617$			
1.56	0.993953383	(2.0384;0.987943327)	(0.00384;0.006010055)
$db\Psi(1,2)=0.006046617$			
2.0384	0.987943327	(2.535936;0.981969612)	(0.535936;0.005973714)
0.0384	0.006010055	(0.559936;0.005973714)	(<0;0.00003634)
$db\Psi(1,3)=0.006082957$			
2.535936	0.981969612	(3.05337344; 0.976032017)	(1.05337344; 0.005937594)

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0.535936	0.005973714	(1.07737344; 0.005937593)	(<0;0.00003612)
0.559936	0.005973714	(0.99833344; 0.005937593)	(<0;0.00003612)
$db\Psi(1,4)=0.006155198$			

Secondly, we suppose that in continuous-time risk model F is Uniform(0,1) distribution function. Using formulae (18)-(19) we obtained values by a process of approximation for ruin probabilities. We give these values[2] in Table 3, Table 4 and Figure 1.

Table 3. The diffusion approximations $\Psi_{DA}(u,\theta)$

$u\backslash\theta$	0.05	0.10	0.20	0.30
1	0.860708	0.740818	0.548812	0.406570
2	0.740818	0.548812	0.301194	0.165299
5	0.472367	0.223130	0.049787	0.011109
10	0.223130	0.049787	0.002479	0.000123
15	0.105399	0.011109	0.000123	0.000001
20	0.049787	0.002479	0.000006	1.5×10^{-8}

Table 4. The corrected diffusion approximations $\Psi_{CDA}(u,\theta)$

$u\backslash\theta$	0.05	0.10	0.20	0.30
1	0.833273	0.701925	0.515883	0.397422
2	0.721372	0.532348	0.310230	0.195053
5	0.467939	0.231497	0.064723	0.019857
10	0.227314	0.057255	0.004338	0.000344
15	0.110340	0.014025	0.000271	0.000005
20	0.053521	0.003409	0.000016	0.7×10^{-7}

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[2] Initial reserve u in monetary units

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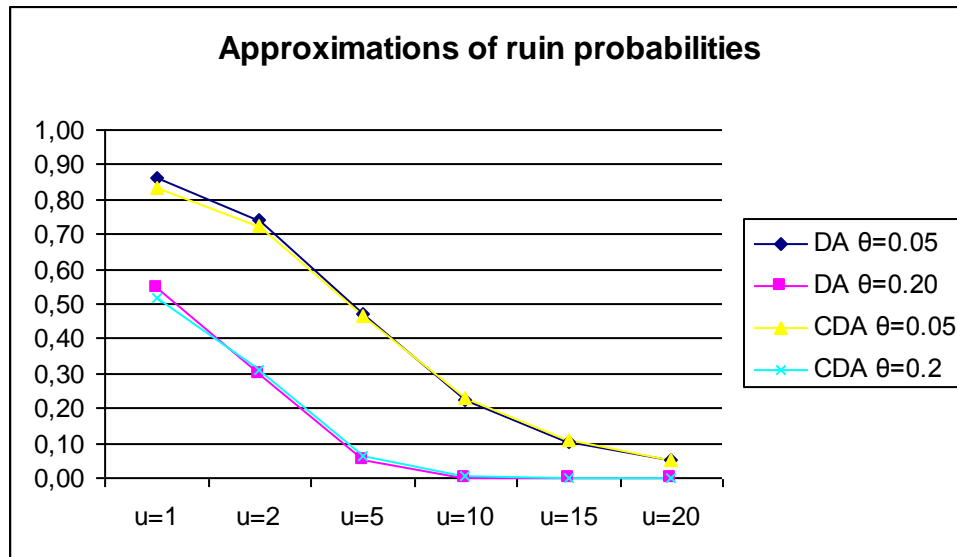
Figure 1. Diffusion approximations $\Psi_{DA}(u,\theta)$ **and** $\Psi_{CDA}(u,\theta)$

Thirdly, we determine the initial reserve for given ruin probability α. For this, we use formulae (24) and (26). Let us consider the individual loss described by the discrete random variable $X_i : \begin{pmatrix} 0 & S \\ q & p \end{pmatrix}$, where $p = 0.03$. We suppose that the number of insurance contracts is $n = 6400$. The values in Table 5 show that for probability of ruin within an accepted domain, the amount of reserves obtained by Chebyshev's inequality are much higher than those obtained by the CLT, tens of times higher.

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Table 5. The amount of the minimum reserves

α	S (m.u.)	θ	$u_{CLT}^{\min}$ (m.u.)	$u_{Ceb}^{\min}$ (m.u.)
0.005	1	0.05	25.5409	183.3972
0.005	1	0.10	15.9409	173,7972
0.005	2	0.05	51.0814	366.7943
0.005	2	0.10	31.8818	347.5943
0.01	1	0.05	22.1292	126.8696
0.01	1	0.10	12.5218	117.2696
0.01	2	0.05	44.2584	253.7392
0.01	2	0.10	25.0584	234.5392

We remark that: i) In the case there is no charge of safety loading (θ=0), the reserve determined either by Chebyshev's inequality or by CLT is unbounded relative to the number claims.

ii) The ratio between the maximum of minimum reserves equals the ratio between the maximum number of claims and depends only on the ruin probability:

$$ratio(\alpha) = \frac{\max u_{Che}^{\min}}{\max u_{CLT}^{\min}} = \frac{1}{\alpha \cdot z_{1-\alpha}^2}.$$

There are applications for which the diffusion approximation delivers poor results, therefore proposed a corrected diffusion approximation that reflects information in the increment distribution beyond the mean and variance. The first problem is to find the expected value of the maximum of a random walk with small, negative drift, and the second problem is to find the distribution of the same quantity. Since Brownian motion is skip-free, the idea to replace the risk process by a Brownian motion ignores the presence of the overshoot and other things. The objective of the corrected diffusion approximation is to take this and other deficits into consideration. Another inconvenient of the diffusion approximation is that θ is close to zero, and we want to consider the given risk process with more safety loading.

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